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SHORT COMMUNICATION
SINGULARITY CANCELLATION IN BOUNDARY INTEGRAL EQUATIONS FOR AXISYMMETRIC STOKES FLOW

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INTRODUCTION

The boundary element method (BEM) provides a scheme for solving boundary integral equations numerically by dividing the boundary into elements and approximating the unknown variables over these elements by means of shape functions. In recent years this method has developed into a powerful tool for solving engineering problems. One difficulty that arises in solving problems by the boundary element method is the computation of integrals that exist only in the sense of the Cauchy principal value. Singular kernels appear in integrals that are on or near the diagonal of the matrices that result from the discretization of the boundary, and the solution is very sensitive to the accurate calculation of these integrals. For axisymmetric creeping flow problems (i.e., at zero Reynolds numbers[1,2]) the singularities are of order $1/r$ and $\log r$. Integration through the logarithmic singularity is easily performed by logarithmic Gauss quadrature.

In Cartesian co-ordinates the calculation of the $1/r$-singular integrals has been overcome by the use of the rigid body motion method.[3] However, in axisymmetric creeping flow, rigid body motion cannot be imposed in the radial direction. Other modes of motion, analogous to rigid body motion, have been suggested by Sarihan and Mukherjee,[4] Bakr[5] and Rizzo and Shippy.[6] For certain problems the principal value of singular integral can even be evaluated analytically.[7] However, because of the complexity of the kernel functions (combinations of elliptic integrals of the first and second kinds), general analytical integration cannot be performed. Recently, Guiggiani and Cassilini[8] reported the direct calculation of Cauchy principal value integrals using standard and logarithmic Gauss quadrature. Telles[9] used a mapping technique to eliminate the singularity. We will show here that these singularities cancel out directly in axisymmetric BEM by means of algebraic simplification for arbitrary elements and mappings. It is more practical to cancel out the $1/r$-singularities and obtain non-singular expressions rather than to calculate them individually using other techniques. We begin by presenting the boundary integral equations that describe creeping flow of an incompressible Newtonian fluid. Then we show how these equations 'simplify' for axisymmetric flow. Finally we show how the singularities cancel for any order elements.

BOUNDARY INTEGRAL EQUATION

Creeping flow of an incompressible Newtonian fluid past a particle with an arbitrary surface Γ is described by

$$\mu\nabla^2\mathbf{u} = \nabla P, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\mathbf{u}, P \rightarrow 0 \quad \text{as } \|\mathbf{y}\| \rightarrow \infty, \qquad \mathbf{u} = \mathbf{u}_{\text{particle}} \quad \text{on } \Gamma, \tag{2}$$

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where **u** is the fluid velocity vector, P is the pressure, μ is the viscosity and **y** is the position vector. The fundamental singular solution of the system (equations (1) and (2)) for a point force at point p in an unbounded fluid is[1,2,10,11]

$$u_{ij}(p, q) = -\frac{1}{8\pi\mu r}\left(\delta_{ij} + \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right), \tag{3}$$

$$t_{ij}(p, q) = \frac{3}{4\pi r^2}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} n_k \frac{\partial r}{\partial x_k}, \tag{4}$$

where r is the distance between points p and q in the fluid, n_k is the kth component of the unit vector normal to the surface and δ_{ij} is the Kronecker delta. Physically, $u_{ij}(p, q)$ is the velocity component in the ith direction at point q resulting from a unit force in the jth direction at point p. The function $t_{ij}(p, q)$ describes the traction field associated with $u_{ij}(p, q)$. That is, $t_{ij}(p, q)$ is the ith component of the traction at point p resulting from a unit force in the jth direction at point q. Note that the fundamental solution is singular when p and q coincide.

For axisymmetric flows the fundamental solution is derived from the general three-dimensional solution by expressing the latter in a cylindrical co-ordinate system. Then the fundamental solution in cylindrical co-ordinates is integrated in the θ-direction around the axis of rotational symmetry. The boundary integral equation becomes

$$c_{ij}(p)u_j(p) = \int_\gamma t_j(q)u_{ij}(p, q)r_q\,\mathrm{d}\gamma - \int_\gamma u_j(q)t_{ij}(p, q)r_q\,\mathrm{d}\gamma, \tag{5}$$

where γ is the contour of the radial section of Γ. The coefficients $c_{ij}(p)$ depend only on the local geometry of the bounding surface Γ at point p. For a smooth surface $c_{ij}(p)$ is equal to $\frac{1}{2}\delta_{ij}$. The point $p(r_p, z_p)$ is fixed and $q(r_q, z_q)$ denotes a boundary point moving along the integration path.

SINGULARITY CANCELLATION

We now present a procedure by which the first-order singularities in the integrals

$$I = \int_\gamma u_j(q)t_{ij}(p, q)r_q\,\mathrm{d}\gamma \tag{6}$$

are removed. These integrals arise from the second term on the right-hand side of equation (5). The Green functions are then obtained as functions of a, b, $\bar{z}$ and $\bar{r}$, where

$$a = r_p^2 + r_q^2 + \bar{z}^2, \qquad b = 2r_pr_q, \tag{7}$$

$$\bar{z} = z_q - z_p, \qquad \bar{r} = r_q - r_p. \tag{8}$$

The Green functions are complicated and are not presented here.[12] The expression for t_{ij} becomes singular when q approaches p, because $a-b$ approaches zero. From equations (7) and (8) we have

$$\frac{1}{a-b} = \frac{1}{\bar{z}^2 + \bar{r}^2} = \frac{1}{r^2}, \tag{9}$$

where r is the distance between p and q. Notice that the singularity can be cancelled out by a factor of the form $\bar{z}^i\bar{r}^{2-i}$ for $i = 0, 1, 2$.

In order to simplify the analytical expression for t_{ij}, it is convenient to isolate the leading singular term of the t_{ij}-expressions and treat it separately from the non-singular parts. This can be done by expressing a and b in terms of $\bar{r}$ and $\bar{z}$ and rearranging the t_{ij} to separate out the non-singular parts where the $1/(a-b)$-term has been cancelled by factors of the form $\bar{z}^i\bar{r}^{2-i}$ with $i=0, 1, 2$. This gives

$$t_{ij} = t'_{ij} + t''_{ij}, \tag{10}$$

where t''_{ij} is the non-singular part (given in the Appendix) and t'_{ij} is the part where $1/(a-b)$ has not been cancelled. The t'_{ij} are given by

$$t'_{ij} = \beta A_{ij} S_{ij} (n_r \bar{r} f^{(2)}_{ij} + n_z \bar{z} f^{(1)}_{ij}), \tag{11}$$

where

$$\beta = \frac{E}{2\pi r_q (a+b)^{3/2}}, \tag{12}$$

n_r and n_z are the components of the outwardly directed unit vector normal to the contour γ and E is an elliptic integral of the second kind. The expressions for A_{ij}, S_{ij}, $f^{(1)}_{ij}$ and $f^{(2)}_{ij}$ are given in Table I. The components of the unit normal are given by

$$n_r = \frac{1}{|J|}\frac{\mathrm{d}z_q}{\mathrm{d}x} = \frac{1}{|J|}\frac{\mathrm{d}\bar{z}}{\mathrm{d}x}, \qquad n_z = -\frac{1}{|J|}\frac{\mathrm{d}r_q}{\mathrm{d}x} = -\frac{1}{|J|}\frac{\mathrm{d}\bar{r}}{\mathrm{d}x}, \tag{13}$$

where J is the Jacobian of transformation from global co-ordinates to local element co-ordinates x. Substituting equation (13) into equation (11), we obtain

$$t'_{ij} = \alpha A_{ij} S_{ij} B_{ij}, \tag{14}$$

where

$$B_{ij} = \bar{r}\frac{\mathrm{d}\bar{z}}{\mathrm{d}x} f^{(2)}_{ij} - \bar{z}\frac{\mathrm{d}\bar{r}}{\mathrm{d}x} f^{(1)}_{ij}, \tag{15}$$

$$\alpha = \frac{\beta}{|J|}. \tag{16}$$

These expressions are general and apply for any order of element and mapping. Note that $|J|$ will be cancelled after the discretization of the radial contour element $\mathrm{d}\gamma$ to local element co-ordinates.

Table I. Definitions of terms appearing in the expressions for the tractions after discretization

	rr	zr	rz	zz	$\theta\theta$
A_{ij}	$\frac{1}{r_p r_q}(r_p+r_q)^2(r_p^2+r_q^2)$	$\frac{1}{r_p}(r_p+r_q)$	r_p+r_q	1	$\frac{a+b}{r_p r_q}$
S_{ij}	$\frac{\bar{r}^2}{(a-b)^2}$	$\frac{\overline{rz}}{(a-b)^2}$	$\frac{\overline{rz}}{(a-b)^2}$	$\frac{\bar{z}^2}{(a-b)^2}$	$\frac{1}{a-b}$
NS_{ij}	$\frac{\Psi_r^2}{(\Psi_r^2+\Psi_z^2)^2}$	$\frac{\Psi_r\Psi_z}{(\Psi_r^2+\Psi_z^2)^2}$	$\frac{\Psi_r\Psi_z}{(\Psi_r^2+\Psi_z^2)^2}$	$\frac{\Psi_z^2}{(\Psi_r^2+\Psi_z^2)^2}$	$\frac{1}{(\Psi_r^2+\Psi_z^2)}$
f^2_{ij}	r_p+r_q	$2(r_p+r_q)(r_p^2+r_q^2)$	$4r_q(r_p+r_q)$	$(r_p+r_q)(r_p^2+7r_q^2)$	$r_p^2(r_p+r_q)$
f^1_{ij}	$2r_q$	$r_q(7r_p^2+r_q^2)$	$r_p^2+7r_q^2$	$8r_q(r_q^2+r_q^2)$	$r_q(r_p^2+r_q^2)$
Δ_{ij}	-1	$r_q^2+3r_pr_q-2r_p^2$	r_p-3r_q	$r_q^2-6r_pr_q+r_p^2$	$r_q^2+r_pr_q+r_p^2$

These results (equations (14) and (15)) are independent of the type of element to be used in the discretization.

We now prove that the singularity cancels out for any element of any order. For a higher-order element the only terms that change in equation (15) are the expressions $\bar{r}\,\mathrm{d}\bar{z}/\mathrm{d}x$ and $\bar{z}\,\mathrm{d}\bar{r}/\mathrm{d}x$. In general, for r_p equal to r_j,

$$\bar{r}=\sum_{i=1}^{N}\phi_i r_i - r_j = r_1\phi_1+\ldots+(\phi_j-1)r_j+\ldots+r_N\phi_N, \tag{17}$$

where ϕ_i are the basis functions of the element and r_i are the r-co-ordinates of the nodes of the element. Also, because $\sum_{j=1}^{N}\phi_j=1$, we have

$$\bar{r}=\sum_{i=1,i\neq j}^{N}\phi_i(r_i-r_j). \tag{18}$$

All the shape functions ϕ_i for $i\neq j$ contain the $(x-x_j)$-term, where x_j is the location of p, because they are equal to zero when $x=x_j$. Therefore we can express ϕ_i for i not equal to j as the common factor $x-x_j$ times a remainder ψ_i:

$$\phi_i=\psi_i(x-x_j). \tag{19}$$

Substituting into equation (18), we obtain

$$\bar{r}=\Psi_r(x-x_j), \qquad \frac{\mathrm{d}\bar{r}}{\mathrm{d}x}=\Psi_r+(x-x_j)\Psi_r', \tag{20}$$

where $\Psi_r=\sum_{i=1,i\neq j}^{N}\psi_i(r_i-r_j)$ and $\Psi_r'=\mathrm{d}\Psi_r/\mathrm{d}x$. Similarly, we obtain

$$\bar{z}=\Psi_z(x-x_j), \qquad \frac{\mathrm{d}\bar{z}}{\mathrm{d}x}=\Psi_z+(x-x_j)\Psi_z', \tag{21}$$

where $\Psi_z=\sum_{i=1,i\neq j}^{N}\psi_i(z_i-z_j)$. From equations (20) and (21) we obtain

$$\bar{z}\frac{\mathrm{d}\bar{r}}{\mathrm{d}x}=(x-x_j)\Psi_z\Psi_r+(x-x_j)^2\Psi_z\Psi_r', \tag{22}$$

$$\bar{r}\frac{\mathrm{d}\bar{z}}{\mathrm{d}x}=(x-x_j)\Psi_z\Psi_r+(x-x_j)^2\Psi_r\Psi_z'. \tag{23}$$

The B_{ij}-terms from equation (15) become

$$B_{ij}=(1+x)\Psi_z\Psi_r(f_{ij}^2-f_{ij}^1)+(1+x)^2(\Psi_r\Psi_z' f_{ij}^{(2)}-\Psi_z\Psi_r' f_{ij}^{(1)}). \tag{24}$$

The difference $f_{ij}^{(2)}-f_{ij}^{(1)}$ always has a common factor of $\bar{r}$:

$$f_{ij}^{(2)}-f_{ij}^{(1)}=\bar{r}\Delta_{ij}. \tag{25}$$

Expressions for Δ_{ij} are given in Table I; the Δ_{ij} are not singular. The common factor $\bar{r}$ in equation (25) gives the additional factor required to cancel the singularity. Thus the B_{ij} would give the required r^2 factor that would cancel the $1/(a-b)$-singularity in S_{ij}.

We have proven that the $1/r$-singularities always cancel out for any order of element. The final non-singular expression for any order of element can be written as

$$t_{ij}'=\alpha A_{ij}NS_{ij}C_{ij}, \tag{26}$$

where

$$C_{ij} = \Psi_r^2 \Psi_z \Delta_{ij} + \Psi_r \Psi'_z f_{ij}^{(2)} - \Psi'_r \Psi_z f_{ij}^{(1)} \tag{27}$$

and the NS_{ij}-expressions are given in Table I. These are non-singular expressions and can be integrated with ease by standard Gauss quadrature.

We have assumed that the mapping is linear. Linear mappings are used predominantly in practice, but higher-order mappings can also be used for curved boundaries. The derivations presented in this communication are the same even for higher-order mappings. The only differences are in the expressions for Ψ_r and Ψ_z.

In this communication we have examined the singularities in the boundary element method for axisymmetric creeping flow. We have shown that the $1/r$-singularity in the integrals involving t_{ij} can be cancelled. The resulting expressions are non-singular and nothing more than standard Gauss quadrature is necessary for their integration.

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APPENDIX

The non-singular parts t''_{ij} of the Green tractions are given by

$$t''_{rr} = (f_1 n_r + g_1 n_z)K + (f_2 n_r + g_2 n_z)E,$$

$$t''_{zr} = (f_2 n_r + g_2 n_z)K + (f_3 n_r + g_3 n_z)E,$$

$$t''_{rz} = (f_4 n_r + g_4 n_z)K + (f_5 n_r + g_5 n_z)E,$$

$$t''_{zz} = (f_5 n_r + g_5 n_z)K + (f_6 n_r + g_6 n_z)E,$$

$$t''_{\theta\theta} = (f_7 n_r + g_7 n_z)K + (f_8 n_r + g_8 n_z)E,$$

where K and E are elliptic integrals of the first and second kinds respectively and f_i and g_i are as follows:

$$f_1 = \frac{1}{r_q^2}\alpha_1\left[\frac{1}{r_p}\bar{z}^4\gamma_z + \left(\frac{2r_q^2}{r_p} + \frac{5}{2}r_p\right)\bar{z}^2\gamma_z + \left(\frac{r_q^4}{2r_p} - \frac{5}{2}r_p r_q^2 + 2r_p^3\right)\gamma_z - \frac{1}{2}\left(\frac{r_q}{r_p} - 1\right)(r_q + r_p)^3\gamma_r\right],$$

$$f_2 = \frac{\alpha_1}{r_q}\left[-\frac{1}{2r_p}\bar{z}^3\gamma_z - \frac{3}{2}\left(\frac{r_q^2}{r_p} + r_p\right)\bar{z}\gamma_z - \frac{1}{r_p}(r_p + r_q)^2\gamma_r\bar{z}\right],$$

$$f_3 = \alpha_1(2r_p^2 - a)\frac{\gamma_z}{2r_p},$$

$$f_4 = \frac{\alpha_1}{4r_q^2}[2\bar{z}^3\gamma_z + (8r_q^2 + 4r_p^2)\bar{z}\gamma_z + 2(r_q + r_p)^2\bar{z}\gamma_r],$$

$$f_5=\frac{\alpha_1}{2r_q}(a-2r_q^2)\gamma_z,$$

$$f_6=-\alpha_1\bar{z}\gamma_z,$$

$$f_7=\frac{-\alpha_1(a+b)^3}{r_p r_q^2}(r_q^2-2a),$$

$$f_8=\frac{\alpha_1(a+b)}{2r_p r_q}\bar{z},$$

$$g_1=\frac{\alpha_1}{r_q^2}\left[-\bar{z}^4\gamma_z^2\frac{1}{r_p}-\left(3\frac{r_q^2}{r_p}+\frac{7}{2}r_p\right)\bar{z}^2\gamma_z^2-\frac{5}{2}r_p(r_p+r_q)^2\gamma_r\gamma_z+\left(-\frac{5}{2}\frac{r_q^4}{r_p}+r_p r_q^2-\frac{9}{2}r_p^3\right)\gamma_z^2\right],$$

$$g_2=\frac{\alpha_1}{r_q}\left(\bar{z}^3\gamma_z^2\frac{1}{2r_p}+\frac{2}{r_p}(r_q^2+r_p^2)\bar{z}\gamma_z^2+(r_p+r_q)^2\frac{5}{2r_p}\gamma_z\gamma_r\bar{z}\right),$$

$$g_3=\frac{\alpha_1}{r_q}\left[\frac{r_q}{2r_p}\bar{z}^2\gamma_z^2+\left(\frac{r_q^3}{r_p}-3r_p r_q\right)\gamma_z^2\right],$$

$$g_4=\frac{\alpha_1}{2r_q^2}\left(-\bar{z}^3\gamma_z^2-(5r_q^2+3r_q^2)\bar{z}\gamma_z^2+(r_q+r_p)(r_q^2+3r_p^2)\gamma_z(\gamma_r\gamma_z)^{1/2}\right),$$

$$g_5=\frac{\alpha_1}{2r_q}[-\bar{z}^2\gamma_z^2+(6r_q^2-2r_p^2)\gamma_z^2],$$

$$g_6=0,$$

$$g_7=\frac{\alpha_1(a+b)^3}{2r_p r_q^2}[2\bar{z}^2\gamma_z+(3r_q^2+4r_p^2)r_q\gamma_z+(r_p+r_q)^2\gamma_r],$$

$$g_8=0,$$

with

$$\alpha_1=\frac{1}{\pi(a+b)^{3/2}},\qquad \gamma_r=\frac{\Psi_r^2}{\Psi_r^2+\Psi_z^2},\qquad \gamma_z=\frac{\Psi_z^2}{\Psi_r^2+\Psi_z^2}.$$

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